LENGTH FUNCTIONS OF 2-DIMENSIONAL RIGHT-ANGLED ARTIN GROUPS

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Abstract. Morgan and Culler proved that a minimal action of a free group on a tree is determined by its translation length function. We prove an analogue of this theorem for two-dimensional right-angled Artin groups acting on CAT(0) rectangle complexes.

1. Introduction

Let G be a finitely generated group and $G \times X \to X$ an isometric action of G on a metric space X. The length function of the action is the function $G \to [0, \infty)$ defined by $l(g) = \inf\{d(x, gx) \mid x \in X\}$. In [8], Culler and Morgan study length functions of groups acting (minimally and semi-simply) on $\mathbb{R}$-trees. They prove that such actions are determined up to equivariant isometry by their length functions. This implies that the space of all such actions, modulo scaling, embeds in an infinite dimensional projective space $\mathbb{P}^\infty$.

In the case of a free group, $G = F_n$, this theorem has important applications. The space of actions of F_n on a simplicial tree, up to scaling, is equivalent to the space of marked graphs introduced by Culler and Vogtmann in [9] in their study of automorphism groups of free groups. This space, which we denote by CV_n, is commonly known as Outer Space. By Culler-Morgan, CV_n embeds in $\mathbb{P}^\infty$. They also prove that the image of this embedding lies in a compact subset of $\mathbb{P}^\infty$, so its closure is a compactification of CV_n, and points on the boundary can be described as "very small" actions of F_n on an $\mathbb{R}$-tree [7]. This has provided an essential tool in the study of automorphism groups of free groups.

In this paper we prove a two-dimensional analogue of Culler and Morgan's theorem. Associated to a finite, simplicial graph Γ with vertex set V is a group A_Γ, called a *right-angled Artin group*, defined by

$$A_\Gamma = \langle V \mid v_i v_j = v_j v_i \text{ if } v_i, v_j \text{ are adjacent in } \Gamma \rangle.$$

At the two extremes, a graph Γ with no edges gives rise to a free group, while a complete graph Γ gives a free abelian group. In recent work with K. Vogtmann, the first author has been studying automorphisms of right-angled Artin groups, [2, 4, 5, 6]. The (outer) automorphism groups $Out(A_\Gamma)$ interpolate between $Out(F_n)$ and $GL_n(\mathbb{Z})$ and provide a context for studying the similarities and differences between these groups. It would be particularly useful to have a good analogue of Outer Space for right-angled Artin groups, as well as a compactification of this space.

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Consider as an example, the case in which A_Γ is a product of two free groups, $F_n \times F_m$. This is the right-angled Artin group associated to the join of two discrete graphs. A natural candidate for Outer Space for this group is the space whose points are actions of A_Γ on a product of two trees. A product of two trees is a CAT(0) rectangle complex, that is, a piecewise Euclidean CAT(0) space whose cells are rectangles. More generally, every right-angled Artin group acts on a rectangle complex, namely its Cayley 2-complex. If Γ has no triangles, then this complex is CAT(0) and the quotient by A_Γ is a $K(A_\Gamma, 1)$-space (see [3]). In this case, we say that A_Γ is 2-dimensional. Thus, for 2-dimensional A_Γ a natural analogue of Culler-Vogtmann's space would be a deformation space of actions on CAT(0) rectangle complexes. (A different notion of Outer Space for these groups was introduced in [5].)

With this in mind, we are interested in an analogue of Culler and Morgan's theorem for 2-dimensional right-angled Artin groups. In this paper we prove the following theorem.

Main Theorem. *Assume Γ has no triangles and no vertices of valence 0. Let X and X' be 2-dimensional CAT(0) rectangle complexes with minimal actions of A_Γ. If the length functions associated to the two actions are the same, then X and X' are equivariantly isometric.*

For free groups acting on trees, an action is said to be "minimal" if there is no invariant subtree, or equivalently, if every edge in the tree lies in the axis of some element. The minimality condition that appears in the theorem above is an analogue of the latter condition. See Section 3 for a discussion of minimality.

The special case of the main theorem in which the complexes are required to be regular cube complexes (i.e. all edge lengths equal 1) appears in the second author's thesis [10].

2. Rectangle complexes

A cubical complex is a piecewise Euclidean complexes all of whose cells are standard Euclidean cubes $C^k = [0,1]^k$. In this paper, we are interested in piecewise Euclidean complexes whose cells are rectangular, that is, each cell is isometric to a finite product of intervals $\prod[0, a_i]$, but the edge lengths may vary from cell to cell. We assume, however, that our complex has finite shapes, that is, there are only finitely many different edge lengths.

Most of the standard properties of cubical complexes hold more generally for rectangle complexes. In particular, the link of a vertex in a rectangle complex Y is a piecewise spherical simplicial complex with all edge lengths $\frac{\pi}{2}$, hence Y is locally CAT(0) if and only if the link of every vertex is flag (i.e., any set of pairwise adjacent vertices in the link spans a simplex). In the case of a 2-dimensional rectangle complex, the link is just a graph and the flag condition is equivalent to the statement that this graph has no 3-cycles. We can also define *walls* in Y, as for cubical complexes, as equivalence classes of midplanes of rectangles. Walls are geodesic and separate Y into two components. A geodesic in Y crosses each wall at most once and a geodesic which intersects a wall in a non-trivial segment must lie totally inside the wall.

Definition 2.1. Let C_1, C_2 be two convex subcomplexes of a CAT(0) rectangle complex Y. A *spanning geodesic* from C_1 to C_2 is a geodesic of minimal length, i.e., whose length is equal to the distance from C_1 to C_2.

If Y has finite shapes, then such a spanning geodesic always exists since the set of distances between rectangles in Y is discrete.

Lemma 2.2. *Let Y be a CAT(0) rectangle complex with finite shapes and let C_1, C_2 be convex sub-complexes.*

(1) *If α is a geodesic in Y from $y_1 \in C_1$ to $y_2 \in C_2$ then α is a spanning geodesic if and only if the angle at y_i between α and C_i is $\geq \frac{\pi}{2}$, for $i = 1, 2$.*
(2) *There exists a spanning geodesic from C_1 to C_2 whose endpoints are vertices of Y.*

Proof. (1) Let α be as above. If the angle at one endpoint, say y_1, between α and C_1 is $< \frac{\pi}{2}$, then we can "cut off the corner" near y_1 to get a shorter path from C_1 to C_2, so α is not a spanning geodesic. Conversely, suppose the angles at both endpoints are $\geq \frac{\pi}{2}$. Let γ be a spanning geodesic with endpoints y_3, y_4. Then y_1, y_2, y_3, y_3 span a quadrilateral in Y with all angles $\geq \frac{\pi}{2}$. By the Flat Quadrilateral Theorem ([1] p. 181), this quadrilateral spans a Euclidean rectangle. In particular, the opposite sides α and γ of this rectangle have the same length.

(2) Let γ be any spanning geodesic from C_1 to C_2 with endpoints y_1, y_2. We proceed by induction on the dimension of Y. If $\dim(Y) = 1$ the lemma is clear. Suppose $\dim(Y) > 1$. Let σ be the smallest face of Y containing the initial segment of γ. If y_1 is not a vertex, then γ is orthogonal to $\sigma \cap C_1$, so the initial segment of γ is parallel to some midplane of σ. It follows that all of γ remains parallel to the wall W containing this midplane and lies in the cellular neighborhood $N(W)$ of this wall (where $N(W)$ is the union of the rectangles with a midplane in W). Note that all edges of X orthogonal to W must have the same length r since parallel edges in a cube have the same length, so $N(W) = W \times [0, r]$. Projecting γ orthogonally onto W, gives another spanning geodesic, so we may assume that, in fact, γ lies in W.

The wall W inherits the structure of a CAT(0) rectangle complex of one dimension lower than X. By induction, there is a spanning geodesic α in W from $C_1 \cap W$ to $C_2 \cap W$ which begins and ends at a vertex of W. Vertices of W correspond to edges of X which meet W orthogonally. The projection $N(W) = W \times [0, r] \to W \times 0$ takes α to a path of the same length which begins and ends at a vertex of X. This is the desired spanning geodesic. □

Lemma 2.3. *Let Y be as above and let C_1, C_2, C_3 be convex sub-complexes. Suppose C_1 and C_2 each intersect C_3 in non-empty sets, but $C_1 \cap C_2 \cap C_3 = \emptyset$. Then any minimal length geodesic in C_3 from $C_1 \cap C_3$ to $C_2 \cap C_3$ is a spanning geodesic for C_1, C_2.*

Proof. Let α be a minimal length geodesic in C_3 from $C_1 \cap C_3$ to $C_2 \cap C_3$ with endpoints p, q. Consider the endpoint p. The minimality of α implies that the distance in $link(p, C_3)$ between α_p and $link(p, C_1 \cap C_3)$ is at least $\frac{\pi}{2}$. Thus, α_p is either a vertex of $link(p, C_3) \backslash link(p, C_1)$, or it lies in an edge of $link(p, C_3)$ neither of whose vertices are in $link(p, C_1)$. In either case, the distance from α_p to $link(p, C_1)$ along *any* path in $link(p, X)$

is at least $\frac{\pi}{2}$. Thus the angle between α and C_1 is $\geq \frac{\pi}{2}$. The same is true at q, so the lemma follows from Lemma 2.2. □

3. Length functions and minimality

Given a group, G, acting by isometries on a metric space (X, d), we define the *translation length function* of the action to be the map $l : G \to \mathbb{R}$ given by

$$l(g) = \inf\{d(x, g.x) | x \in X\}.$$

An important concept in the study of translation length functions is the notion of a *minset.* The minset of $g \in G$ is the set on which $l(g)$ is realized, that is,

$$min(g) = \{x \in X | d(x, g.x) = l(g)\}.$$

The action of G on X is said to be *semi-simple* if for all $g \in G$, $min(g) \neq \emptyset$. An element of G is a *hyperboic* isometry, if its minset is non-empty and its translation length is non-zero. In particular, if the action of G is free and semi-simple, then every element is hyperbolic.

More generally, for a subgroup, $H < G$, the minset of H is defined by

$$min(H) = \bigcap_{g \in H} min(g).$$

For actions on CAT(0) spaces, the structure of minsets is well understood and detailed in [1]. Notably, g is hyperbolic if and only if there exists a geodesic line in X on which g acts as a non-trivial translation. Such a line is called an *axis* of g and $l(g)$ is equal to the translation length along this axis. The minset of g decomposes as a product $min(g) = Y \times \mathbb{E}^1$ where Y is a convex subspace fixed by g each line $\{y\} \times \mathbb{E}^1$ is an axis for g. Moreover generally, by the Flat Torus Theorem, if $H < G$ is isomorphic to $\mathbb{Z}^k$, then $min(H)$ is isometric to $Y \times \mathbb{E}^k$ for some convex subspace Y, H fixes Y and acts on $\mathbb{E}^k$ by translations.

Now suppose A_Γ is a right-angled Artin group whose defining graph Γ has no triangles and no components consisting of a single vertex (i.e., no valence 0 vertices). We are interested in actions of A_Γ on 2-dimensional CAT(0) rectangle complexes. In this situation, the maximal rank of an abelian subgroup is 2 and for such a subgroup H, the minset of a H is just a single flat, $\mathbb{E}^2$ (since Y is convex and 0-dimensional). The rectangular structure on this flat gives an orthogonal grid and H acts as a finite index subgroup of the group of translations of this grid. In particular, H contains elements which translate parallel to each of the two grid directions. We call such an element a *gridline isometry.*

As in the case of free groups acting on trees, we will need a concept of "minimality" for our actions. For a free group acting semi-simply on tree, a minimal action is defined as one for which there is no invariant subtree. This is equivalent to requiring that every point in the tree lie in the minset of some group element, that is, every point lies on some axis. This property is key to the proof of Culler and Morgan's theorem. With this in mind, we define

Definition 3.1. Assume Γ is has no triangles and no valence 0 vertices. Suppose A_Γ acts properly, semi-simply by isometries on a CAT(0) rectangle complex X. We will say that

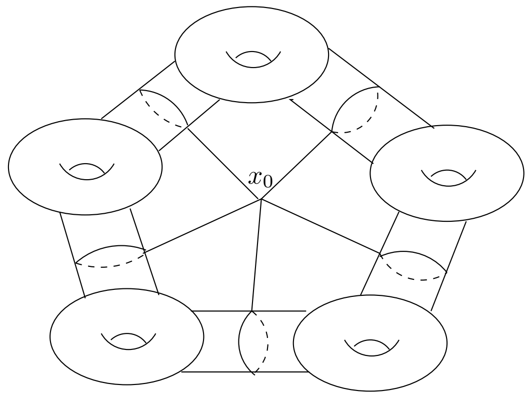

FIGURE 1.

the action of A_Γ is *minimal* if X is 2-dimensional (so minset of $\mathbb{Z}^2$-subgroups are isometric to $\mathbb{E}^2$) and X is covered by the minsets of its $\mathbb{Z}^2$-subgroups.

We remark that the minimality condition implies that the action is cocompact and hence also that X has finite shapes. Also, since A_Γ is torsion-free, any proper action is a free action, so every element of A_Γ is hyperbolic.

It is easy to see that minimality implies that X has no convex invariant subspace. Unlike the tree case, however, the converse is not true as the next example shows.

Example 3.2. Let Γ be a pentagon with edges $e_1, \ldots, e_5$. Construct a cubical complex $\bar{X}$ as follows (see Figure 1.) Start with 5 disjoint tori $T_1, \ldots, T_5$ corresponding to the 5 edges of Γ. For each v vertex of Γ, glue a tube of length 2 between the two tori containing curves marked v. The tubes create a non-trivial loop of length 10 in the center. Glue a pentagon made of 5 squares onto this loop. Call the central vertex of this pentagon x_0. Then it is easy to check that the fundamental group of $\bar{X}$ is A_Γ and the links in $\bar{X}$ are flag. It follows that the universal covering space X of $\bar{X}$ is CAT(0) and has a free, cocompact action of A_Γ. It is also easy to see that X is complete and has the geodesic extension property, so by Lemma 6.20 of [1], X has no A_Γ-invariant convex subcomplex. On the other hand, it does not satisfy our definition of minimality since the link of any vertex lying over x_0 is a circle of radius $\frac{5\pi}{2}$ hence cannot lie in any flat.

4. INTERSECTIONS OF MINSETS

From now on, we assume that Γ has no triangles and no valence 0 vertices. Suppose X is a 2-dimensional CAT(0) rectangle complex with a minimal A_Γ action. Our goal is to prove that the geometry of X and the action of A_Γ is determined by the length function. In light of the definition of minimality, the interaction between minsets of $\mathbb{Z}^2$-subgroups of A_Γ will be central to the proof of the theorem.

By definition of A_Γ, any pair of adjacent vertices in Γ generates a $\mathbb{Z}^2$-subgroup. More generally, if $J = V_1 * V_2$ is a complete bipartite subgraph of Γ, then A_J is a product of free

groups, $A_J = F(V_1) \times F(V_2)$, so any pair of non-trivial elements $h_1 \in F(V_1)$, $h_2 \in F(V_2)$ generates a $\mathbb{Z}^2$-subgroup.

Definition 4.1. We say that a subgroup $H < A_\Gamma$ is a *basic $\mathbb{Z}^2$-subgroup* if H is a maximal $\mathbb{Z}^2$-subgroup and H is contained in $A_J = F(V_1) \times F(V_2)$ for some complete bipartite subgraph $J \subset \Gamma$.

If H is a basic subgroup, it is a simple exercise to show that there is a pair of generators h_1, h_2 for H, unique up to inversion, such that $h_1 \in F(V_1)$ and $h_2 \in F(V_2)$. We refer to these as *basic generators* of H.

Lemma 4.2. *Every maximal $\mathbb{Z}^2$-subgroup H is conjugate to a basic $\mathbb{Z}^2$-subgroup and has a unique (up to inversion) set of generators conjugate to the basic generators. (We call these the basic generators of H.)*

Proof. The first assertion follows from results of Servatius [11] which we recall here briefly. An element of A_Γ is called cyclically reduced if it is of minimal length in its conjugacy class. For any $g \in A_\Gamma$, there exists a unique cyclically reduced element conjugate to g. Thus, for any $\mathbb{Z}^2$-subgroup $H < A_\Gamma$, we may assume up to conjugacy that H contains a cyclically reduced element h.

In [11], Servatius describes explicitly the centralizer of a cyclically reduced element. Write $Supp(h)$ for the set of generators appearing in a minimal length word for h. It follows from Servatius' theorem that the centralizer of h is cyclic unless the subgraph of Γ spanned by $Supp(h)$ decomposes as a non-trivial join, or the set L of vertices commuting with all of $supp(h)$ is non-empty, in which case $Supp(h) \cup L$ span a join. Since H is contained in the centralizer of h, this centralizer is non-cyclic and the first assertion follows.

The uniqueness of basic generators follows from the fact that the normalizer of any $\mathbb{Z}^2$-subgroup is equal to its centralizer. □

We can now prove some simple facts about $\mathbb{Z}^2$-minsets.

Lemma 4.3. *Let H_1, H_2 be $\mathbb{Z}^2$-subgroups with minsets M_1 and M_2. Suppose $M_1 \cap M_2 \neq \emptyset$. Then one of the following holds.*

1. *$H_1 \cap H_2 = \{1\}$ and $M_1 \cap M_2$ is a compact (possibly degenerate) rectangle.*
2. *$H_1 \cap H_2 = \langle h \rangle$, $h \neq 1$, and $M_1 \cap M_2$ is an infinite line or infinite Euclidean strip consisting of a union of h-axes.*

Proof. First note that $M_1 \cap M_2$ is a convex subcomplex of M_i. If $H_1 \cap H_2$ contains a non-trivial element h, then $M_1 \cap M_2$ is h-invariant, hence it is either empty or it is a union of h-axes. Conversely, if $M_1 \cap M_2$ is unbounded, then it contains a ray α along some gridline. Both H_1 and H_2 contain gridline isometries h_1, h_2 that translate along this ray. The fact that the action is proper then implies that $h_1^n = h_2^m$ for some n and m, so $H_1 \cap H_2$ is non-trivial. □

Lemma 4.4. *Suppose H_1, H_2 are $\mathbb{Z}^2$-subgroups with $H_1 \cap H_2 = \langle h \rangle$, $h \neq 1$. Then their minsets M_1, M_2 have branching along some h axis. In particular, h is a gridline isometry.*

Proof. If $M_1 \cap M_2 \neq \emptyset$, this follows from the previous lemma. If $M_1 \cap M_2 = \emptyset$, it follows from the Flat Quadrilateral Theorem ([1], p. 181) that the set of spanning geodesics between them forms a convex, h-invariant Euclidean strip. This strip intersects each M_i in a single h-axis. □

The structure of links at branch points in X will be key to the proof of the main theorem. We are particularly interested in branch points whose links are sufficiently complicated.

Definition 4.5. A point $x \in X$ is a *corner* if branching occurs at x along a pair of orthogonal grid lines, or equivalently, if $link(x, X)$ is not isometric to the suspension of some discrete set.

The terminology comes from the following observation. If x is a corner as defined above, then for any minset M containing x, we can find minsets M_1 and M_2 (not necessarily distinct) such that x is a corner, in the usual sense, of the rectangle $M \cap M_1 \cap M_2$.

Certain minsets must have corners. For example, if v, w are adjacent vertices of valence at least 2 in Γ then it follows from Lemma 4.4 that the minset if $\langle v, w\rangle$ has branching along both the v and w axes, hence it must have corners. Also, if Γ has more than one component, then there is some $\mathbb{Z}^2$-subgroup in each component whose minset has corners. To see this, note that since X is connected, some minset from each component intersects a minset in some other component. This intersection must be compact, hence gives rise to corners.

On the other hand, if Γ is the star of a single vertex v, then no minset in X has corners since in this case. $X = min(v) = \mathbb{E}^1 \times T$ and all branching occurs along some axis of v. As the following lemma shows, however, in all other cases, X has plenty of corners.

Lemma 4.6. *Assume Γ is not the star of a single vertex and suppose $H < F(V_1) \times F(V_2)$ is a basic $\mathbb{Z}^2$-subgroup whose minset M has no corners. Then*

1. *either V_1 or V_2 consists of a single vertex of valence at least 2, and*
2. *M is contained in a union of $\mathbb{Z}^2$-minset with corners.*

Proof. We may assume without loss of generality that $V_1 * V_2$ is a maximal join in Γ. Let $H = \langle h_1, h_2\rangle$ be a basic generating set for H with $h_i \in F(V_i)$. If V_1 and V_2 both contain at least 2 elements, then we can choose $v_i \in V_i$ such that $v_i \neq h_i$. Letting $H_1 = \langle h_1, v_2\rangle$ and $H_2 = \langle v_1, h_2\rangle$, it follows from Lemma 4.4 that M has branching along both h_1 and h_2 axes. This contradicts the assumption that M has no corners.

Thus one of the sets V_i consists of a single vertex, say $V_1 = \{v\}$ and V_2 consists of all the adjacent vertices. If V_2 also contains only a single vertex w, then by the maximality of $F(V_1) \times F(V_2)$, the edge between v and w is a component of Γ. Since Γ is not a star, it must have additional components and it follows from the discussion above that M has corners. This again contradicts our assumption, so V_2 must have cardinality at least 2.

To prove the second statement of the lemma, consider the minset of v. It decomposes as $\mathbb{R} \times T$ where the first factor is an axis for v and the second factor is a tree on which the free group $F(V_2)$ acts. For any $g \in F(V_2)$, let $\alpha(g)$ denote the axis for g in T. Then $min\langle v, g\rangle$ is the product $R \times \alpha(g) \subset \mathbb{R} \times T$. We claim that at least one of these minset has

corners. If some $w \in V_2$ has valence at least 2, this follows from the discussion preceding the lemma. If every $w \in V_2$ has valence 1, then $st(v) = V_1 * V_2$ is an entire component of Γ. Since by hypothesis, Γ is not the star of a single vertex, it must contain other components and the claim follows from the first paragraph of the proof.

Say $H = \langle v, h \rangle$, $h \in F(V_2)$. Choose $p \in F(V_2)$ such that $M' = min\langle v, p\rangle$ has corners. Set $g = h^k p^k$ for some fixed k and consider the axes for p, h, and g in T. It follows from basic facts about trees, that for k sufficiently large, $\alpha(g) \cap \alpha(p)$ is a segment of length $> l(p)$ (i.e., its p-translates cover $\alpha(p)$), and $\alpha(g) \cap \alpha(h)$ is a segment of length $> l(h)$ (i.e., its h-translates cover $\alpha(h)$). From this we conclude that $min\langle v, g\rangle$ intersects M' in an infinite strip containing corners, and it intersect M is an infinite strip whose h-translates cover M. The second statement of the lemma follows. $\square$

5. Distances between minsets

In this section we will show that the length function determines the distances between $\mathbb{Z}^2$-minsets.

Theorem 5.1. *Let G, H be maximal $\mathbb{Z}^2$-subgroups of A_Γ with minsets M_G, M_H.*

(1) *If $M_G \cap M_H \neq \emptyset$, then $l(gh) \leq l(g) + l(h)$ for all $g \in G, h \in H$.*
(2) *If $M_G \cap M_H = \emptyset$, then the distance d between M_G and M_H satisfies*

$$2d = \sup\{l(gh) - l(g) - l(h) \mid g \in G, h \in H\} > 0.$$

We begin by establishing some notation.

- For a geodesic segment ϕ from x_1 to x_2 write $\phi = [x_1, x_2]$ and $\overline{\phi} = [x_2, x_1]$.
- For an element $w \in A_\Gamma$, let $\phi^w = [wx_1, wx_2]$, the translate of ϕ by w.
- If $\phi = [x_1, x_2]$ and $\psi = [x_2, x_3]$, denote the piecewise geodesic $\phi \cdot \psi$ by $[x_1, x_2, x_3]$.
- Denote by ϕ_{x_i} the tangent vector to ϕ at x_i, viewed as a point in $link(x_i, X)$.
- For two geodesic segments $\alpha = [x, y]$ and $\beta = [x, z]$, the *angle between α and β* is the distance between α_x and β_x in $link(x, X)$. (In particular, our angles can be greater than π.)

To motivate the proof of the theorem, let us recall what happens in the case of a free group F acting on a tree T. If the axes for two elements $g, h \in F$ do not intersect, then the distance between them is exactly $\frac{1}{2}[l(gh) - l(g) - l(h)]$. To prove this, one notes that the spanning geodesic γ between the axes can be extended geodesically in either direction along the two axes. It follows that if x, y are the endpoints of γ, then the piecewise geodesic $[g^{-1}x, x, y, hy, hx, ghx]$ is, in fact, geodesic (see Figure 2). Since the last segment $[hx, ghx]$ is the gh-translate of the first segment $[g^{-1}x, x]$, translating this geodesic by powers of (gh), we obtain a gh-invariant line, namely an axis for gh. The translation length along this axis is now easily seen to be $l(g) + l(h) + 2d$, where $d =$length(γ).

To imitate this argument in the 2-dimensional setting, we can again start with a spanning geodesic γ between a pair of non-intersecting minsets M_G and M_H with endpoints at vertices. If X is a standard cubical complex with all side lengths equal 1, then any geodesic between vertices crosses each cube at a rational slope and in every $\mathbb{Z}^2$-minset, lines of rational slope are axes of some element. In this situation, we can extend γ geodesically,

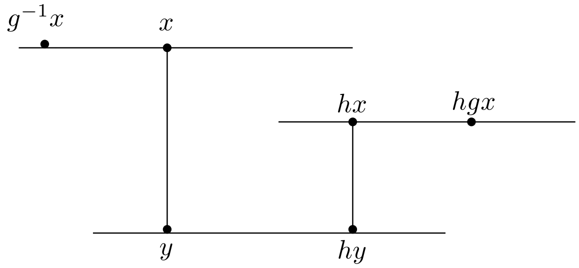

FIGURE 2. Axis of gh in a tree

as in the tree case, along axes for some $g \in G$ and $h \in H$ to obtain an axis for gh with translation length $l(g) + l(h) + 2d$ (see [10]).

In the case of a general rectangle complex, the situation is more complicated. Since the shapes of the cubes can differ as we move from one minset to another, the directions corresponding to axes will not necessarily be rationally related. Thus, we cannot always extend γ geodesically along the axis of some element of G or H. Nonetheless, as we will see below, we can approximate the tree picture. Namely, for $g \in G$ and $h \in H$, consider the piecewise geodesic $[g^{-1}x, x, y, hy, hx, ghx]$. Taking translates by powers of gh gives a gh-invariant piecewise geodesic $\phi(g, h)$, which we will call the *piecewise axis* for the pair g, h. We will prove that for any $\epsilon > 0$, one can choose g and h such that this piecewise axis has Hausdorff distance less than ϵ from a true gh-axis (i.e., they lie within ϵ-neighborhoods of each other).

The proof proceeds as follows. We begin by developing $\phi(g, h)$ onto the plane. We then show that for appropriate choices of g and h, we can "straighten" this piecewise geodesic in the plane so that the resulting line corresponds to an axis for gh in X with the desired property.

To develop the piecewise axis onto the plane, we need to understand the structure of the links at the endpoints of a spanning geodesic. Let M be a 2-flat in X, $x \in M$, and $C = lk(x, M)$. Then C is a circle of length 2π in $lk(x, X)$. Let $t \in lk(x, X)$ be a point at distance at least $\frac{\pi}{2}$ from C, and let Θ be the smallest subgraph of $lk(x, X)$ which contains C and t. Note that Θ need not be connected.

Lemma 5.2. *Let Θ_1 and Θ_2 be the graphs in Figure 3. Then Θ is isomorphic to either Θ_1 or to a subgraph of Θ_2 (some or all of the dotted edges may be missing).*

Proof. If t lies in the interior of an edge e whose endpoints are connected to antipodal points in C as in Θ_1, then Θ must be isomorphic to Θ_1. This is because no other edges can be attached to these vertices without creating a 3-cycle. Otherwise, either one endpoint of e is not connected to C by an edge, or the endpoints are connected to a pair of adjacent vertices. In this case, Θ must be a subgraph of Θ_2. (For example, if the distance from t to C is greater than π, then Θ consists of just C and e, so all of the dotted edges are missing.)

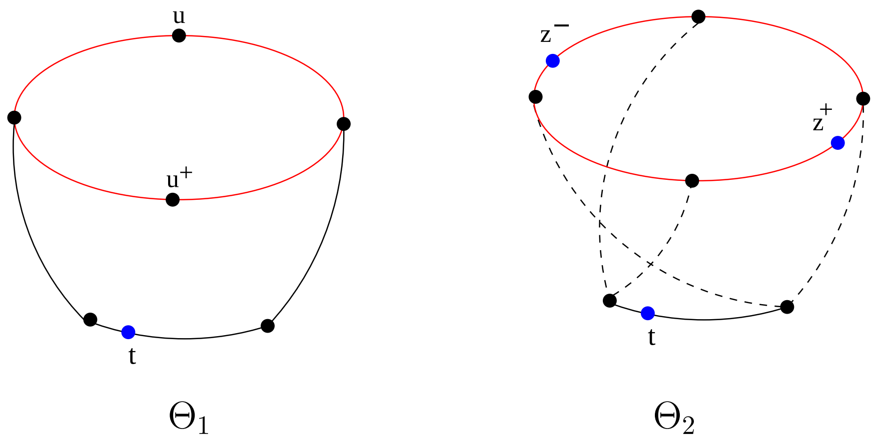

FIGURE 3. Span of t and C in $link(x, X)$

In case t is itself a vertex, Θ contains only C and the edges (if any) connecting t to C. In this case Θ is again isomorphic to a subgraph of Θ_2. □

Suppose ρ is a geodesic segment in X which is transverse to every edge it crosses. Then any segment of ρ not containing a vertex has a neighborhood isometric to a Euclidean strip. This strip can be continued across a vertex v if and only if the incoming and outgoing tangent vectors $\rho_v^{\pm}$ lie in a circle of radius 2π, i.e. a 4-cycle, in $link(v, X)$. On the other hand, if they are not contained in a 4-cycle, then the tangent vectors can be perturbed toward a "missing edge" of the 4-cycle and still remain distance $\geq \pi$ apart. In other words, ρ can be "bent" at v in at least one direction and still remain geodesic. We think of such a missing edge as a "slit" in the Euclidean strip about ρ, and we will talk about bending ρ towards the slit.

Now let γ be a spanning geodesic between two $\mathbb{Z}^2$-minsets $M_G = min(G)$ and $M_H = min(H)$ with endpoints at vertices $x \in M_G$ and $y \in M_H$. Let $\gamma_1 = [x, x_1]$ (respectively $\gamma_2 = [y_2, y]$) be the longest initial (respectively final) segment of γ which has a neighborhood in X isometric to a Euclidean strip. Note that it is possible (for example if γ lies in a single minset) that $\gamma_1 = \gamma_2 = \gamma$. In this case we say γ is *unbendable*. Otherwise, it is *bendable* and we write $\gamma = \gamma_1\gamma_0\gamma_2$. The middle segment, γ_0, may reduce to a single point if $x_1 = y_1$.

Ideally, we would like to find an isometry $g \in G$ such that at x, both rays of the g-axis through x form geodesic extensions of γ, or in other words, the pair of antipodal points in $link(x, X)$ corresponding to the g-axis are distance at least π from γ_x. Let $t = \gamma_x$ and note that since γ is a spanning geodesic, t is distance at least $\frac{\pi}{2}$ from $C = lk(x, M_G)$. Define Θ, as above, to be the smallest subgraph of $lk(x, X)$ which contains C and t. Then by Lemma 5.2, Θ is isomorphic to either Θ_1 or a subgraph of Θ_2. We say that x is of *type 1* or *type 2* accordingly. In the latter case, we choose an identification of Θ with a subgraph of Θ_2 and call the edges of Θ_2 not contained in Θ *fictitious edges*.

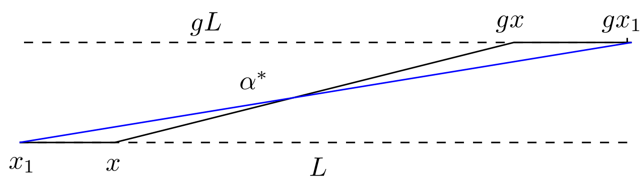

FIGURE 4. α^* with no slits

If x is of type 1, or if t is a vertex, then there is a pair of antipodal vertices on C, $u^{\pm}$, at distance at least π from t. Choosing g to be a gridline isometry in direction $u^{\pm}$, gives an axis satisfying the desired condition, so if α is the geodesic from x to gx, then $\overline{\gamma} \cdot \alpha \cdot \gamma^g = [y, x, gx, gy]$ is geodesic.

However, if x is of type 2 and t is not a vertex, there are only two possible pairs of antipodal points at distance π from t in Θ_2 and there is no guarantee that these pairs correspond to the axis of some $g \in G$. The best we can do is to choose g with axis close these antipodal points.

Lemma 5.3. *For $g \in G$, let α denote the geodesic from x to gx in M_G. Then for any $\epsilon > 0$, there exists $g \in G$ such that the Hausdorff distance between the geodesic $\alpha^* = [x_1, gx_1]$ and the piecewise geodesic $\overline{\gamma_1} \cdot \alpha \cdot \gamma_1^g = [x_1, x, gx, gx_1]$ is less than ϵ. Moreover, the angles between α^* and this piecewise geodesic at the endpoints are also less than ϵ.*

Proof. As observed above, if x is of type 1 of $t = \gamma_x$ is a vertex, we can take g so that $\alpha^* = \overline{\gamma_1} \cdot \alpha \cdot \gamma_1^g$. So assume that x is of type 2 and t lies in the interior of some edge of *theta*. Choose a pair of antipodal points $z^{\pm}$ in C at distance at π from t. These points are the tangent vectors to a line L in M_G.

By the Dirichlet's Approximation Theorem, we can find a lattice point gx (i.e., a point in the G-orbit of x) arbitrarily close to L. Moreover, this point can be taken to be arbitrarily far from x, thus the angle between α and L at x is arbitrarily small, as is the angle between α and gL at gx, i.e. the tangent vectors to α lie arbitrarily close to $z^{\pm}$.

We wish to develop the piecewise geodesic $\overline{\gamma_1} \cdot \alpha \cdot \gamma_1^g$ onto the plane so that the planar angles at x and gx are less than or equal to the actual angles in X. Each of the 3 geodesic segments lies in a Euclidean strip, so the key is to determine how to assemble these strips at x and gx. For this, we use the identification of Θ with (a subgraph of) Θ_2. Namely, we think of the 4-cycle spanned by t and z^+ as the link of x in the plane, and the 4-cycle spanned by t and z^- as the link of gx in the plane. Note that the lines L and gL are parallel in M_G, so the segments $[x_1, x]$ and $[gx, gx_1]$ appear as parallel segments when developed onto the plane (see Figure 4).

Fictitious edges, if any, are indicated by slits (see Figure 5). Note that slits at x are caused by missing edges in the 4-cycle spanned by t and z^+ whereas slits at gx are caused

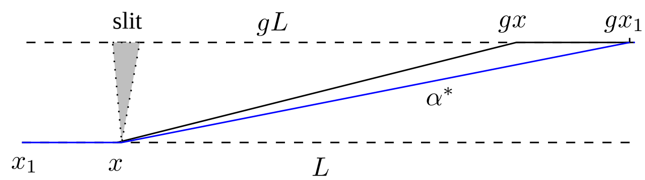

FIGURE 5. α^* with a slit at x

by missing edges in the 4-cycle spanned by t and z^-. Thus, it is possible to have slits at one of these points but not the other. The planar angle measured across a slit is strictly less than the corresponding distance in the link of x (or gx) in X.

The strip between L and gL in the plane, thus corresponds to a slit Euclidean strip in X containing $\overline{\gamma}_1 \cdot \alpha \cdot \gamma_1^g$. If $\overline{\gamma}_1 \cdot \alpha \cdot \gamma_1^g$ bends toward a slit, then it is already locally geodesic at that point. If not, then it can be straightened to a geodesic α^* in X without leaving this strip. The lemma follows. $\square$

Note that we can choose g so that gx lies on either side of L. In particular, if γ is bendable, so there are slits at x_1 and gx_1, we can choose g so that straightening $\overline{\gamma}_1 \cdot \alpha \cdot \gamma_1^g$ bends it toward the slit and hence, $\overline{\gamma}_0 \cdot \alpha^* \cdot \gamma_0^g$ is still geodesic.

We can also apply the lemma at the other endpoint y of γ to find $h \in H$ with geodesic β from y to hy such that the geodesic $\beta^* = [y_1, hy_1]$ is ϵ-close to $\gamma_2 \cdot \beta \cdot \overline{\gamma}_2^h = [y_1, y, hy, hy_1]$ and if γ is bendable, then $\gamma_0 \cdot \beta^* \cdot \overline{\gamma}_0^h$ is geodesic. Putting these together we conclude

Proposition 5.4. *If γ is bendable, then there exist $g \in G, h \in H$ such that*

$$(\beta^*)^{h^{-1}} \cdot \overline{\gamma}_0 \cdot \alpha^* \cdot \gamma_0^g \cdot (\beta^*)^g \tag{1}$$

is geodesic. Its gh-translates form a gh-axis that lies within Hausdorff distance ϵ of the piecewise axis

$$\phi(g,h) = \cdots \beta^{h^{-1}} \cdot \overline{\gamma} \cdot \alpha \cdot \gamma^g \cdot \beta^g \cdots \tag{2}$$

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. First assume that $M_G \cap M_H \neq \emptyset$ and let x be a point in this intersection. Then for any $g \in G$, $h \in H$, $l(gh) \leq d(x, ghx) \leq d(x, gx) + d(gx, ghx) = l(g) + l(h)$. This proves the first statement of the theorem.

Now assume $M_G \cap M_H = \emptyset$ and let d be the distance between them. Let γ be a spanning geodesic from M_G to M_H whose endpoints x, y are vertices. We first note that for any

$g \in G$, $h \in H$,

$$\begin{aligned} l(gh) &\leq d(x, ghx) \\ &\leq d(x, gx) + d(gx, gy) + d(gy, ghy) + d(ghy, ghx) \\ &= l(g) + l(h) + 2d. \end{aligned}$$

To prove the theorem, it suffices to show that for any $\epsilon > 0$, we can find g, h such that the piecewise axis $\phi(g, h)$ lies within ϵ of some true axis for gh. For in this case, each segment of $\phi(g, h)$ has length at most 2ϵ more than its projection on the axis, so

$$l(g) + l(h) + 2d \leq l(gh) + 8\epsilon.$$

If γ is bendable, then this follows from Proposition 5.4. If γ is unbendable, but both endpoints of γ are of type 1, then the piecewise axis $\phi(g, h)$ is already geodesic, hence it is an axis for gh.

So assume from now on that γ is unbendable and that x is of type 2. If y is type 1, then $\gamma \cdot \beta \cdot \overline{\gamma}^h$ is geodesic and has angles at y and hy strictly greater than π. by Lemma 5.3, g can be chosen so that the geodesic α^* from y to gy makes arbitrarily small angles with $\overline{\gamma}$ and γ^g. It follows that $\beta^{h^{-1}} \cdot \alpha^* \cdot \beta^g$ is geodesic and its translates form an axis for gh satisfying the desired condition.

There remains the case that both links are of type 2. Arguing as in the proof of Lemma 5.3, we can develop $\overline{\gamma} \cdot \alpha \cdot \gamma^g$ onto the plane so that $\overline{\gamma}$ and γ^g appear as parallel segments and lie in a Euclidean strip of width ϵ. Moreover, if there is a slit at x, we can choose g such that the bending at x is towards the slit. Since y is also of type 2, we can do the same for $\gamma^{h^{-1}} \cdot \beta^{h^{-1}} \cdot \overline{\gamma}$ and hence likewise for $\gamma^g \cdot \beta^g \cdot \overline{\gamma}^{gh}$. Continuing this process we can develop the entire piecewise axis $\phi(g, h)$ onto the plane. Let $\tilde{\phi}$ denote the image of $\phi(g, h)$ in the plane.

The translates of γ all appear parallel in $\tilde{\phi}$ and $\tilde{\phi}$ is invariant under the translation of the plane which takes γ to γ^{gh}. It follows that the convex hull of $\tilde{\phi}$ is a strip of width at most 2ϵ bounded by a pair of parallel lines L_1 and L_2. See Figure 6. This strip corresponds to a gh-invariant strip E, possibly with slits, in X. We will show that E contains an axis for gh.

If there are no slits at any vertex of $\tilde{\phi}$, then L_1 and L_2 lift to axes for gh in X. Suppose there are slits at some vertices. We may assume (by appropriate choice of g and h) that $\tilde{\phi}$ bends toward the slit at some vertex. Among all such vertices, choose one, call it v, closest to a bounding line L_1 or L_2. (In Figure 6, for example, if both gx and gy had inward pointing slits, then we would choose $v = gx$.) Say v is closest to L_1 with the slit pointing downward. Let L_v be the straight line in the plane through v and ghv.

Consider the geodesic from v to ghv in X. It's image in the plane is the shortest path from v to ghv which does not cross any slit. This path lies between L_v and L_2. It follows that the gh-translates of this path bend (if at all) toward the slit at v and translates of v. Hence they lift to a gh-invariant geodesic in E, i.e., an axis. $\square$

In the course of the proof we have shown

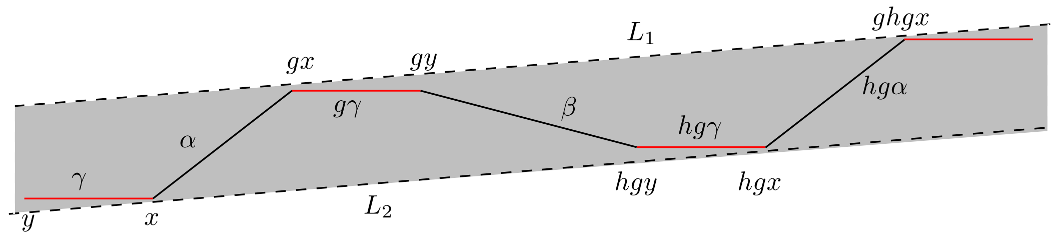

FIGURE 6. The piecewise axis $\phi(g,h)$ developed onto the plane.

Theorem 5.5. *Let G,H be maximal $\mathbb{Z}^2$-subgroups with minsets M_G, M_H and suppose $M_G \cap M_H = \emptyset$. Let γ be a spanning geodesic from M_G to M_H with endpoints at vertices. Then for any $\epsilon > 0$, there exists $g \in G, h \in H$ such that some axis for gh intersects γ and lies within Hausdorff distance ϵ of the piecewise axis $\phi(g,h)$.*

For a pair of minsets with $M_G \cap M_H \neq \emptyset$, their intersection is a (possibly degenerate) rectangle or an infinite Euclidean strip. We will refer to all of these as "rectangles" of side length $r_1, r_2 \in [0,\infty]$. The following key fact is an easy consequence of the theorem.

Corollary 5.6. *If $M_G \cap M_H$ is a non-empty rectangle of side lengths $r_1, r_2 \in [0,\infty]$, then the dimensions r_1, r_2 are determined by the length function. If either of the r_i satisfy $r_i > 0$, then the length function also determines which gridline isometries in G and H act in the same direction along this side of the rectangle.*

Proof. Let $R = M_G \cap M_H$. First note that R is a single point if and only if for every $g \in G$, $l(g) = d(M_H, gM_H)$. By Theorem 5.1, the right hand side is determined by the length function.

So assume from now on that at least one of the r_i is non-zero. In this case, we claim that the length function determines which elements $g \in G$ act as gridline isometries. If R is an infinite strip, then $G \cap H = \langle h \rangle$ and h is a gridline isometry in both minsets. The orthogonal gridline isometries g are then determined by the lengths of $l(g), l(h), l(g^{-1}h)$.

Now assume R is compact, and note that for any $g \in G$ with $R \cap gR = \emptyset$, $d(R, gR) = d(M_H, gM_H)$ is determined by the length function. Next note that

$$l(g) + d(R, gR) \leq d(R, g^2R)$$

with equality holding if and only if the axis for g passes through two corners of R (see Figure 7), and in this case, the distance between the two corners is $l(g) - d(R, gR)$. There are at most 4 (parallel classes of) such axes in M_G, two of which are along gridlines, namely the two for which $l(g) - d(R, gR)$ is smallest. It follows that the length function determines which elements of G act along gridlines as well as the side lengths of R along these gridlines. The same holds for H.

If $r_1 \neq r_2$, this determines the identification of the rectangle R in M_G with the rectangle R in M_H up to reflection. To obtain the correct orientation, consider gridline isometries $g \in G$ and $h \in H$ which translate along a side of positive length length $r = r_1$ or r_2. If h

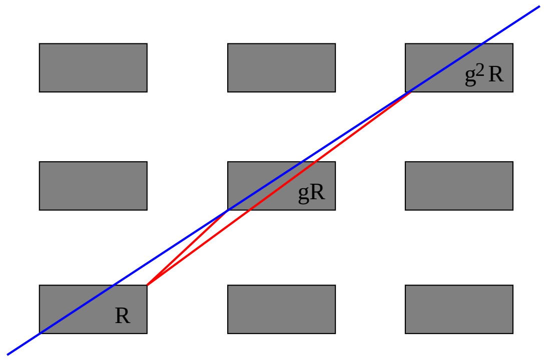

FIGURE 7. The orbit of R in M_G

translates in the same direction as g then

$$d(gM_H, hM_G) = d(gR, hR) = d(gR, R) + d(R, hR) = l(g) + l(h) - 2r$$
$$d(gM_H, h^{-1}M_G) = d(gR, h^{-1}R) = d(gR, R) + d(R, h^{-1}R) + r = l(g) + l(h) - r$$

so we can distinguish between the directions h and h^{-1} relative to g.

Finally, if $r_1 = r_2$, then the gridline isometries h and g act in the same direction if and only if the above two equations are satisfied and, in addition, for a gridline isometry $k \in H$ orthogonal to h,

$$d(gM_H, k^{\pm 1}M_G) \leq d(gR, k^{\pm 1}R) \leq d(gR, R) + d(R, k^{\pm 1}R) = l(g) + l(k) - 2r$$

In particular, we must have $d(gM_H, h^{-1}M_G) > d(gM_H, hM_G) \geq d(gM_H, k^{\pm 1}M_G)$. This completes the proof of the second statement. □

6. Main theorem

We are now ready to prove our main theorem.

Theorem 6.1. *Assume Γ has no triangles and no vertices of valence 0. Let X and X' be 2-dimensional CAT(0) rectangle complexes with minimal actions of A_Γ. If the length functions associated to the two actions are the same, then X and X' are equivariantly isometric.*

Proof. If Γ is the star of a single vertex v, then $A_\Gamma \cong \mathbb{Z} \times F$ with $\mathbb{Z} = \langle v \rangle$ and F is the free group generated the remaining vertices. In this case, $X = min(v) = \mathbb{E}^1 \times T$ where the first factor is an axis for v and the second factor T is a tree. It follows from the Flat Torus Theorem ([1], Thm II.7.1) that the action of F on X descends to an action ρ of F on T such that for $(r, t) \in X$, $g \in F$,

$$g \cdot (r, t) = (r + \lambda(g),\ \rho(g)(t))$$

where $\lambda : F \to \mathbb{R}$ is some homomorphism. It is easy to see that the minimality of A_Γ acting on X is equivalent to the minimality of F acting on T, so by Culler and Morgan

[8], it suffices to show that the length function of A_Γ acting on X determines both the homomorphism λ and the length function l_ρ of F acting on T. For this, fix $g \in F$ and consider the action of $H = \langle v, g\rangle$ on its minset M. Pick a basepoint $x_0 \in M$ and identify $M = \mathbb{E}^1 \times \alpha(g) = \mathbb{R}^2$ with x_0 as the origin. Consider the triangle formed by x_0, vx_0, gx_0. Note that the coordinates of gx_0 in $\mathbb{R}^2$ are precisely $(\lambda(g), l_\rho(g))$. This triangle is uniquely determined by the three lengths $l(v), l(g), l(gv)$ (and the fact that v translates in a positive direction along the first factor), hence so are the coordinates of gx_0.

Now assume that Γ is not the star of a single vertex. Fix a maximal $\mathbb{Z}^2$-subgroup H and let M, M' be the minsets of H in X and X' respectively. For any other minset M_i intersecting M, let $R_i = M \cap M_i$ and $R'_i = M' \cap M'_i$.

We first prove that if M has corners, then there is a unique isometry $\phi_H : M \to M'$ such that ϕ_H maps R_i onto R'_i for all i. If some R_i is compact, then by Corollary 5.6 and Theorem 5.1, the shape of R_i and the distances between its H-translates are determined by the length function. So there is a unique H-equivariant isometry ϕ_H from M to M' that identifies R_i with R'_i. For any other rectangle (or strip) R_j, the position of R_j in M is uniquely determined by its distance from the grid of rectangles formed by the H-translates of R_i. By Lemma 2.3, the distance from R_j to such a translate hR_i is equal to the distance from M_j to hM_i so it is determined by the length function. It follows that ϕ_H takes R_j to R'_j for all j.

If R_i is an infinite strip for every M_i, then the same is true for R'_i by Lemma 4.3. Since we are assuming that M has corners, there exist such strips in both gridline directions. By Corollary 5.6 and Theorem 5.1, the direction (horizontal or vertical) of these strips, their widths, and the distances between them are all determined by the length function hence there is a unique H-equivariant isometry ϕ_H from M to M' that identifies each R_i to R'_i.

We have defined an isometry $\phi_H : M \to M'$ for every $\mathbb{Z}^2$-minset with corners, such that all of these maps agree on overlaps. By Lemma 4.6, these minsets cover X, thus the ϕ_H fit together to give a map $\Phi : X \to X'$. The uniqueness of ϕ_H implies that the combined map Φ is A_Γ-equivariant. Moreover, the induced maps $link(x, M) \to link(\phi_H(x), M')$ likewise fit together to give an isomorphism of graphs $link(x, X) \to link(\Phi(x), X')$. It follows that Φ takes local geodesics to local geodesics. Since X and X' are CAT(0), we conclude that Φ is a global isometry. This completes the proof of the theorem. $\square$

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